SINC-COLLOCATION METHOD WITH ORTHOGONALIZATION FOR SINGULAR POISSON-LIKE PROBLEMS

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ABSTRACT. This paper uses the Sinc-collocation method to solve singular Poisson-like problems (a first- or higher-order partial derivative of the exact solution is unbounded on the boundary). A linear system is obtained which is the same as that obtained by using the Sinc-Galerkin method. With a smart choice of the stepsize and the number of the gridpoints, the orthogonalization technique is successfully applied to solve the linear system obtained, and a numerical approximation is obtained with an exponential accuracy $O(\exp(-cN^{\frac{1}{2}}))$, where N is a truncation parameter and c is a constant independent of N.

1. INTRODUCTION

We first consider the two-dimensional Poisson problem (PP):

(1.1)
$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = f(x,y), & (x,y) \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where the domain $\Omega \equiv (a_1, b_1) \times (a_2, b_2)$, and where the first- or higher-order partial derivative of the exact solution is unbounded on the boundary. The standard finite difference or finite element method will experience some difficulties with the above singular PP [12]. To overcome the difficulties, Stenger [9, 10] and Lund et al. [3, 4] used the Sinc-Galerkin method to solve PP by finding the inverse of the matrix associated with the Sylvester equation, and thus obtained the numerical solutions with an accuracy of exponential order $O(\exp(-cN^{\frac{1}{2}}))$.

In this paper, we use the Sinc-collocation method with the orthogonalization technique to establish a scheme to solve PP and obtain a numerical solution with an accuracy of the same exponential order $O(\exp(-cN^{\frac{1}{2}}))$.

In §2 we review some basic facts about the Sinc approximation and derive useful Sinc-collocation formulas in two dimensions.

In §3 we apply the Sinc-collocation method for PP to construct a numerical scheme and obtain the Sylvester equation, which is the same as that obtained by using an entirely different procedure (the Sinc-Galerkin method) [4]. Then we prove that if a smart choice is made of the stepsize and the gridpoint number, the Sylvester equation can be solved by an orthogonalization technique, which permits a significant reduction in both storage and computation compared with

what would be required by traditional methods. Some numerical results are given in the same section.

In §4 we consider a Poisson-like problem:

$$\begin{cases} \dfrac{\partial^4 u}{\partial x^4} + \dfrac{\partial^4 u}{\partial y^4} = f(x, y), & (x, y) \in \Omega \equiv (a_1, b_1) \times (a_2, b_2), \\ u|_{\partial\Omega} = u_x|_{\partial\Omega} = u_y|_{\partial\Omega} = 0, \end{cases}$$

and prove that this kind of problem can also be solved by the same technique described in §3. Some numerical examples are also given in this section.

2. SINC APPROXIMATION

2.1. Notation and background. The goal of this section is to recall notation and definitions of the Sinc function, state some known results, and derive useful formulas that are important for this paper.

First we denote the set of all integers, the set of all real numbers, and the set of all complex numbers by $\mathbf{Z}$, $\mathbf{R}$, and $\mathbf{C}$, respectively.

Notation 2.1. (1) $\operatorname{sinc}(z) = \sin(\pi z)/\pi z$, $z \in \mathbf{Z}$.

Note that $|\operatorname{sinc}(x)| \leq 1$ for any $x \in \mathbf{R}$.

(2) $S(n, h)(z) = \operatorname{sinc}((z - nh)/h)$, $z \in \mathbf{C}$, $h > 0$.

(3) $C(f, h, x) = \sum_{k=-\infty}^{\infty} f(kh) S(k, h)(x)$, $h > 0$.

Here, $C(f, h, x)$ is called the Whittaker cardinal function of $f(x)$ whenever this series converges.

(4) $C_N(f, h, x) = \sum_{k=-N}^{N} f(kh) S(k, h)(x)$.

Definition 2.1. Let $d > 0$, and let $\mathscr{D}_d$ denote the region $\{z = x + iy : |y| < d\}$ in the complex plane, and ϕ the conformal map of a simply connected domain $\mathscr{D}$ in the complex plane onto $\mathscr{D}_d$ such that $\phi(a) = -\infty$ and $\phi(b) = \infty$, where a and b are boundary points of $\mathscr{D}$, i.e., $a, b \in \partial\mathscr{D}$. Let ψ denote the inverse map of ϕ, and let the arc Γ, with endpoints a and b $(a, b \notin \Gamma)$, be given by $\Gamma = \psi(-\infty, \infty)$. For $h > 0$, let the points x_k on Γ be given by $x_k = \psi(kh)$, $k \in \mathbf{Z}$.

For example, if the finite interval is (a, b), take

$$\phi(x) = \ln \frac{x - a}{b - x},$$

and

$$(2.1) \qquad \psi(w) = \frac{a + be^w}{1 + e^w}, \qquad x_k = \frac{a + be^{kh}}{1 + e^{kh}}.$$

Definition 2.2. Let $B(\mathscr{D})$ denote the family of all functions f that are analytic in $\mathscr{D}$ such that for x real,

$$\int_{\psi(x+L)} |f(z)\, dz| \to 0 \qquad (x \to \infty),$$

where $L = \{iy : -d \leq y \leq d\}$, and such that

$$\mathscr{N}(f, \mathscr{D}) = \liminf_{C \to \partial\mathscr{D}, C \subset \mathscr{D}} \int_C |f(z)\, dz| < \infty,$$

where C is a simple closed contour in $\mathscr{D}$.

The following important theorems were proved by Stenger [7].

Theorem 2.1. *If $f \in B(\mathscr{D})$, then*

$$\|f(x) - C(f, h, x)\|_\infty \leq \frac{\mathscr{N}(f, \mathscr{D})}{2\pi d \, \sinh(\pi d/h)} \leq \frac{2\mathscr{N}(f, \mathscr{D})}{\pi d} e^{-\pi d/h},$$

provided $h \leq 2\pi d / \ln 2$.

Theorem 2.2. *Let $f \in B(\mathscr{D})$, and let f satisfy $|f(x)| < Ce^{-\alpha|x|}$ for all $x \in \mathbf{R}$, where C and α are positive constants. Then, by choosing $h = \sqrt{\pi d/\alpha N} \leq 2\pi d/\ln 2$, we have*

$$(2.2) \qquad \|f(x) - C_N(f, h, x)\|_\infty \leq C_1 \sqrt{N} \exp(-\sqrt{\pi d \alpha N}),$$

where C_1 is a constant depending only on f and α.

If x is on the curve Γ, we obtain the following theorem by introducing the conformal map ϕ.

Theorem 2.3. *Let $\phi' f \in B(\mathscr{D})$. Then*

$$(2.3) \qquad \left| f(x) - \sum_{k=-\infty}^{\infty} f(x_k) S(k, h) \circ \phi(x) \right| \leq \frac{\mathscr{N}(f\phi', \mathscr{D})}{2\pi d \, \sinh(\pi d/h)}$$

$$\leq \frac{2\mathscr{N}(f\phi', \mathscr{D})}{\pi d} e^{-\pi d/h}, \qquad x \in \Gamma.$$

Moreover, if $|f(x)| \leq Ce^{-\alpha|\phi(x)|}$, $x \in \Gamma$, for some positive constants C and α, and if the selection $h = \sqrt{\pi d/\alpha N} \leq 2\pi d/\ln 2$ is made in (2.3), then

$$\left| f(x) - \sum_{k=-N}^{N} f(x_k) S(k, h) \circ \phi(x) \right| \leq C_2 \sqrt{N} \exp(-\sqrt{\pi d \alpha N}), \qquad x \in \Gamma,$$

where C_2 depends only on f, d and α.

2.2. Approximation of derivatives on Γ. As indicated by Stenger in [9], the formula (2.3) is not useful for accurately approximating derivatives of f on Γ, except on $\mathbf{R}$, since the terms $\phi' S(k, h) \circ \phi$ are unbounded on Γ. Hence, a "nullifier" function g is introduced to get a formula for approximating $f^{(m)}$ on Γ.

Let g be an analytic function defined on $\mathscr{D}$, and for $k \in \mathbf{Z}$ set

$$(2.4) \quad S_k(z) = g(z) \, \mathrm{sinc} \left[\frac{\phi(z) - kh}{h} \right] = g(z) S(k, h) \circ \phi(z), \qquad z \in \mathscr{D}.$$

The following theorem has been established in [5]. Since part of the proof of this theorem will be used later, the proof is included here.

Theorem 2.4. *Let m be a nonnegative integer, and let $\phi' f/g \in B(\mathscr{D})$. Let there exist positive constants α, C_2 depending only on m, d, and g, a constant C_1*

depending only on m and g, and a constant C_0 depending only on g and f, such that

$$(2.5) \qquad \left| \frac{f(x)}{g(x)} \right| \le C_0 e^{-\alpha|\phi(x)|}, \qquad x \in \Gamma,$$

$$(2.6) \qquad \left| \left(\frac{d}{dx} \right)^n S_k(x) \right| \le C_1 h^{-n} \quad \forall x \in \Gamma, \ n = 0, 1, \dots, m,$$

$$(2.7) \qquad \left| \left(\frac{d}{dx} \right)^n \left\{ \frac{g(x)\sin[\pi\phi(x)/h]}{\phi(z) - \phi(x)} \right\} \right|$$
$$\le C_2 h^{-n} \quad \forall x \in \Gamma, \ z \in \partial\mathscr{D}, \ n = 0, 1, \dots, m.$$

Then there exists a constant K depending only on m, d, α, g, and f such that if $h = \sqrt{\pi d / \alpha N}$, then

$$\left| f^{(n)}(x) - \sum_{j=-N}^{N} \frac{f(x_j)}{g(x_j)} S_j^{(n)}(x) \right| \le K N^{\frac{n+1}{2}} \exp(-\sqrt{\pi d\alpha N})$$

for all $x \in \Gamma$, and for $n = 0, 1, \dots, m$, where $\phi(x_j) = jh$.

Proof. It is easy to show from the residual theorem that if $\phi' f / g \in B(\mathscr{D})$, then for each $x \in \Gamma$,

$$(2.8) \qquad \begin{aligned} f(x) - \sum_{j=-\infty}^{\infty} \frac{f(x_j)}{g(x_j)} S_j(x) &= f(x) - \sum_{j=-\infty}^{\infty} \frac{f(x_j)}{g(x_j)} g(x) S(j, h) \circ \phi(x) \\ &= \frac{g(x)\sin[\pi\phi(x)/h]}{2\pi i} \int_{\partial\mathscr{D}} \frac{[f(z)\phi'(z)/g(z)]\,dz}{[\phi(z) - \phi(x)]\sin[\pi\phi(z)/h]}. \end{aligned}$$

Differentiating both sides of (2.8) n times with respect to x, using (2.7), and noting that $|\Im\phi(z)| = d$ for all $z \in \partial\mathscr{D}$ so that $|\sin[\pi\phi(z)/h]| \ge \sinh(\pi d/h)$, we find that

$$(2.9) \qquad \begin{aligned} &\left| f^{(n)}(x) - \sum_{j=-\infty}^{\infty} \frac{f(x_j)}{g(x_j)} S_j^{(n)}(x) \right| \\ &\le \frac{1}{2\pi} \int_{\partial\mathscr{D}} \left| \left(\frac{d}{dx} \right)^n \left\{ \frac{g(x)\sin[\pi\phi(x)/h]}{\phi(z) - \phi(x)} \right\} \right| \left| \frac{f(z)\phi'(z)/g(z)}{\sin[\pi\phi(z)/h]} \, dz \right| \\ &\le \frac{C_2 h^{-n}}{2\pi \sinh(\pi d/h)} \int_{\partial\mathscr{D}} \left| \frac{f(z)\phi'(z)}{g(z)} \, dz \right| \\ &\le \frac{2C_2 h^{-n}}{\pi} e^{-\pi d/h} \int_{\partial\mathscr{D}} \left| \frac{f(z)\phi'(z)}{g(z)} \, dz \right|, \end{aligned}$$

since $1/\sinh(\pi d/h) < 4e^{-\pi d/h}$ provided $h < 2\pi d / \ln 2$.

Now from the triangle inequality we have that

$$\left| f^{(n)}(x) - \sum_{j=-N}^{N} \frac{f(x_j)}{g(x_j)} S_j^{(n)}(x) \right|$$

$$\leq \left| f^{(n)}(x) - \sum_{j=-\infty}^{\infty} \frac{f(x_j)}{g(x_j)} S_j^{(n)}(x) \right| + \left| \sum_{|j|>N} \frac{f(x_j)}{g(x_j)} S_j^{(n)}(x) \right|.$$

Recalling that $\phi(x_j) = jh$ and applying (2.5) and (2.6), we get that

$$\left| \sum_{|j|>N} \frac{f(x_j)}{g(x_j)} S_j^{(n)}(x) \right| \leq \sum_{|j|>N} C_0 e^{-\alpha|\phi(x_j)|} C_1 h^{-n}$$

$$= \frac{2C_0 C_1}{h^n} \sum_{j=N+1}^{\infty} e^{-\alpha jh} \leq \frac{2C_0 C_1}{\alpha h^{n+1}} e^{-\alpha Nh}.$$

Hence, the above inequality with (2.9) gives

(2.10)
$$\left| f^{(n)}(x) - \sum_{j=-N}^{N} \frac{f(x_j)}{g(x_j)} S_j^{(n)}(x) \right|$$

$$\leq \frac{2 C_2 h^{-n}}{\pi} e^{-\pi d/h} \int_{\partial \mathscr{D}} \left| \frac{f(z)\phi'(z)}{g(z)} \, dz \right| + \frac{2 C_0 C_1}{\alpha h^{n+1}} e^{-\alpha Nh}.$$

Since $\phi(\partial \mathscr{D}) = \partial \mathscr{D}_d$, the integral $\int_{\partial \mathscr{D}} \left| \frac{f(z)\phi'(z)}{g(z)} \, dz \right|$ is a constant depending only on f, g, and d. The proof is then completed if we set $h = \sqrt{\pi d/\alpha N}$ and $K = \max\{K_n, n = 0, 1, \ldots, m\}$, where

$$K_n = \frac{2 C_2}{\pi} \int_{\partial \mathscr{D}} \left| \frac{f(z)\phi'(z)}{g(z)} \, dz \right| \left(\frac{\alpha}{\pi d} \right)^{\frac{n}{2}} + \frac{2 C_0 C_1}{\alpha} \left(\frac{\alpha}{\pi d} \right)^{\frac{n+1}{2}}. \quad \square$$

It should be noted that the function g takes on different forms for different functions ϕ; usually, $g(x) = (\phi'(x))^{-\beta}$ is satisfactory, which was verified by Lundin and Stenger in [5] for the cases of approximation over $(-1, 1)$ and $(0, \infty)$.

2.3. Extension of the approximation to two dimensions. For the sake of simplicity, we consider in the rest of this section only the two-dimensional case, although the following procedure works for any dimension. We will develop some useful results of the Sinc-collocation approximation for two dimensions.

We set $\mathbf{x} = (x, y)$. Later, a bar over some letter always means that this letter refers to the y-coordinate.

Given $\mathbf{d} = (d_1, d_2) \in \mathbf{R}^2$ with $d_i > 0$ for $i = 1, 2$ and conformal maps $\psi : \mathscr{D}_{d_1} \to \mathscr{D}_1$ and $\bar{\psi} : \mathscr{D}_{d_2} \to \mathscr{D}_2$, set $\mathscr{D}_{\mathbf{d}} = \mathscr{D}_{d_1} \times \mathscr{D}_{d_2}$ and $\mathscr{D} = \mathscr{D}_1 \times \mathscr{D}_2$, define $\underline{\psi} : \mathscr{D}_{\mathbf{d}} \to \mathscr{D}$ in coordinate-wise fashion, so $\underline{\psi} = \psi \times \bar{\psi}$. Set $\Gamma_1 = \psi(\mathbf{R})$ and $\Gamma_2 = \bar{\psi}(\mathbf{R})$, and set $\Gamma = \underline{\psi}(\mathbf{R}^2) = \Gamma_1 \times \Gamma_2$.

Furthermore, set, as usual, $\underline{\phi} = \underline{\psi}^{-1}$. Given $\mathbf{h} = (h_1, h_2) \in \mathbf{R}^2$ with positive components h_i $(i = 1, 2)$, and given $\mathbf{k} \in \mathbf{Z}^2$, define $(x_{k_1}, y_{k_2}) =$

$(\psi(k_1 h_1), \bar{\psi}(k_2 h_2))$ and

$$S_k(x) = g(x)S(k, h_1)\phi(x), \qquad \bar{S}_k(y) = \bar{g}(y)\bar{S}(k, h_2)(y),$$

where $k \in \mathbf{Z}$.

We will note some analogues of the previous theorems as they apply to the complex functions $F : \mathscr{D} \to \mathbf{C}$. Let

$$\mathscr{D}_1' = \mathscr{D}_1 \times \Gamma_2, \qquad \mathscr{D}_2' = \Gamma_1 \times \mathscr{D}_2.$$

We say that $F \in B(\mathscr{D}_j')$ if $F \in B(\mathscr{D}_j)$ by fixing the remaining variable.

Analogous to Theorem 2.4, it is not difficult to get

Lemma 2.1. (1) *If* $f(x, y)\phi'(x)/g(x) \in B(\mathscr{D}_1')$, *then for any* $(x, y) \in \Gamma$, *we have*

$$f(x, y) - \sum_{k=-\infty}^{\infty} \frac{f(x_k, y)}{g(x_k)} S_k(x)$$

$$= \frac{g(x)\sin[\pi\phi(x)/h_1]}{2\pi i} \int_{\partial \mathscr{D}_1} \frac{f(z, y)\phi'(z)/g(z)}{[\phi(z) - \phi(x)]\sin[\pi\phi(z)/h_1]} \, dz.$$

(2) *Let* $f(x, y)\phi'(x)/g(x) \in B(\mathscr{D}_1')$. *Assume*

$$\left| \left(\frac{d}{dx}\right)^\ell \left\{ \frac{g(x)\sin[\pi\phi(x)/h_1]}{\phi(z) - \phi(x)} \right\} \right| \le C_{1,2} h_1^{-\ell}$$

for any $x \in \Gamma_1$, $z \in \partial \mathscr{D}_1$, *and* $h_1 < 2\pi d_1/\ln 2$. *Then*

$$\left| \left(\frac{d}{dx}\right)^\ell \{f(x, y)\} - \sum_{k=-\infty}^{\infty} \frac{f(x_k, y)}{g(x_k)} \left(\frac{d}{dx}\right)^\ell \{S_k(x)\} \right|$$

$$(2.11) \qquad \le \frac{C_{1,2} h_1^{-\ell}}{2\pi \sin[\pi d_1/h_1]} \int_{\partial \mathscr{D}_1} \left| \frac{f(z, y)\phi'(z)}{g(z)} \, dz \right|$$

$$\le \frac{2C_{1,2}}{\pi} h_1^{-\ell} e^{-\frac{\pi d_1}{h_1}} \int_{\partial \mathscr{D}_1} \left| \frac{f(z, y)\phi'(z)}{g(z)} \, dz \right|.$$

(3) *If all the conditions in* (2) *are satisfied and*

$$\left| \frac{f(x, y)}{g(x)} \right| \le C_0 e^{-\alpha_1 |\phi(x)|} \quad \forall(x, y) \in \Gamma,$$

and

$$\left| \left(\frac{d}{dx}\right)^\ell \{S_k(x)\} \right| \le C_{1,1} h_1^{-\ell} \quad \forall x \in \Gamma_1,$$

then

$$\left| \left(\frac{d}{dx}\right)^\ell f(x, y) - \sum_{k=-N_j}^{N_j} \frac{f(x_k, y)}{g(x_k)} \left(\frac{d}{dx}\right)^\ell \{S_k(x)\} \right|$$

$$\le \frac{2C_{1,2}}{\pi} h_1^{-\ell} e^{-\frac{\pi d_1}{h_1}} \int_{\partial \mathscr{D}_1} \left| \frac{f(z, y)\phi'(z)}{g(z)} \, dz \right| + \frac{2C_0 C_{1,1}}{\alpha_1 h_1^{\ell+1}} e^{-\alpha_1 N_1 h_1}.$$

A similar formula can be obtained for the y-coordinate.

We always assume later $\Gamma_1 = (a_1, b_1)$ and $\Gamma_2 = (a_2, b_2)$ to be finite, so

$$(2.12) \qquad \phi(x) = \ln \frac{x - a_1}{b_1 - x}, \qquad g(x) = \frac{(x - a_1)^{\beta_1}(b_1 - x)^{\beta_1}}{(b_1 - a_1)^{\beta_1}}, \qquad \beta_1 > 0,$$

$$(2.13) \qquad \bar{\phi}(y) = \ln \frac{y - a_2}{b_2 - y}, \qquad \bar{g}(y) = \frac{(y - a_2)^{\beta_2}(b_2 - y)^{\beta_2}}{(b_2 - a_2)^{\beta_2}}, \qquad \beta_2 > 0.$$

It then follows that

$$(2.14) \qquad\qquad |g(x)| \leq \frac{(b_1 - a_1)^{\beta_1}}{4^{\beta_1}} \quad \text{for } a_1 < x < b_1,$$

$$(2.15) \qquad\qquad |\bar{g}(y)| \leq \frac{(b_2 - a_2)^{\beta_2}}{4^{\beta_2}} \quad \text{for } a_2 < y < b_2.$$

Lemma 2.2. *Let $\phi(x)$ and $g(x)$ be defined as in (2.12) and $\beta_1 \geq \ell$, where ℓ is a nonnegative integer. If $h_1 > 0$ is uniformly bounded by h^*, then*

$$|S_k^{(\ell)}(x)| = \left| \left(\frac{d}{dx} \right)^\ell \{g(x)S(k, h_1) \circ \phi(x)\} \right|$$

$$\leq C_0 h_1^{-\ell}(x - a_1)^{\beta_1 - \ell}(b_1 - x)^{\beta_1 - \ell} \leq K h_1^{-\ell} \quad \forall x \in \Gamma_1,$$

where C_0 and K depend only on β_1, ℓ, a_1, b_1, and h^*.

Proof. The proof is almost the same as that of Theorem 6.3 in [5] by a translation argument. $\square$

Remark 2.1. It is interesting to note that the constant K does not depend on $k \in \mathbf{Z}$, which means that the condition $|S_0^{(\ell)}(x)| \leq K h_1^{-\ell}$ for all $x \in \Gamma_1$ implies $|S_k^{(\ell)}(x)| \leq K h_1^{-\ell}$ for all $x \in \Gamma_1$ and all $k \in \mathbf{Z}$. There is an analogous result for $|\bar{S}_k^{(\ell)}(y)|$.

Lemma 2.3. *Let $f(x, y)$ satisfy*

(1)

$$\left| \frac{f(x, y)}{g(x)\bar{g}(y)} \right| \leq C_0 e^{-\alpha_1|\phi(x)|} e^{-\alpha_2|\bar{\phi}(y)|}, \qquad (x, y) \in \mathscr{D},$$

(2)

$$\frac{f(x, y)\phi'(x)}{g(x)} \in B(\mathscr{D}_1'), \qquad \frac{f(x, y)\bar{\phi}'(y)}{\bar{g}(y)} \in B(\mathscr{D}_2'),$$

(3)

$$\left| \left(\frac{d}{dx} \right)^n \left\{ \frac{g(x)\sin[\pi\phi(x)/h_1]}{\phi(z) - \phi(x)} \right\} \right| \leq C_1 h_1^{-n} \quad \text{for all } x \in \Gamma_1, \ z \in \partial\mathscr{D}_1,$$

where $C_1 = C_1(n, g, a_1, b_1)$, and

$$\left| \frac{\bar{g}(y)\sin[\pi\bar{\phi}(y)/h_2]}{\bar{\phi}(z) - \bar{\phi}(y)} \right| \leq C_2 \quad \text{for all } y \in \Gamma_2, \ z \in \partial\mathscr{D}_2,$$

where $C_2 = C_2(\bar{g}, a_2, b_2),$

(4)
$$|S_0^{(n)}(x)| \le C_4 h_1^{-n} \quad \forall x \in \Gamma_1, \qquad |\bar{S}_0(y)| \le C_5 \quad \forall y \in \Gamma_2,$$

where $C_4 = C_4(\beta_1, n, a_1, b_1, h^*)$ *and* $C_5 = C_5(\beta_2, a_2, b_2, h^*)$.

Let $h_j = \sqrt{\pi d_j / \alpha_j N_j} \le h^*$, $j = 1, 2$. Then there exists a constant C depending on $g, \bar{g}, n, \beta_j, a_j,$ and b_j $(j = 1, 2)$ such that

$$\left| \left(\frac{d}{dx} \right)^n f(x, y) - \sum_{k_1=-N_1}^{N_1} \sum_{k_2=-N_2}^{N_2} \frac{f_{k_1 k_2}}{g_{k_1} \bar{g}_{k_2}} S_{k_1}^{(n)}(x) \bar{S}_{k_2}(y) \right|$$

$$\le C N_1^{\frac{n+1}{2}} \left(E_1 |\bar{g}(y)| e^{-\alpha_2 |\bar{\phi}(y)|} + N_2^{\frac{1}{2}} E_2 \right),$$

where $f_{k_1 k_2} = f(x_{k_1}, y_{k_2})$, $g_{k_1} = g(x_{k_1})$, $\bar{g}_{k_2} = \bar{g}(y_{k_2})$, *and* $E_j = e^{-\sqrt{\pi d_j \alpha_j N_j}}$.

Proof. We can get

$$\left| \left(\frac{d}{dx} \right)^n f(x, y) - \sum_{k_1=-N_1}^{N_1} \sum_{k_2=-N_2}^{N_2} \frac{f_{k_1 k_2}}{g_{k_1} \bar{g}_{k_2}} S_{k_1}^{(n)}(x) \bar{S}_{k_2}(y) \right| \le J_1 + J_2,$$

where

$$J_1 = \left| \left(\frac{d}{dx} \right)^n f(x, y) - \sum_{k_1=-N_1}^{N_1} \frac{f(x_{k_1}, y)}{g_{k_1}} S_{k_1}^{(n)}(x) \right|,$$

$$J_2 = \left| \sum_{k_1=-N_1}^{N_1} \frac{S_{k_1}^{(n)}(x)}{g_{k_1}} \left[f(x_{k_1}, y) - \sum_{k_2=-N_2}^{N_2} \frac{f(x_{k_1}, y_{k_2})}{\bar{g}_{k_2}} \bar{S}_{k_2}(y) \right] \right|.$$

Using Lemma 2.1(3), we have

$$J_1 \le \frac{2C_1}{\pi} h_1^{-n} e^{-\frac{\pi d_1}{h_1}} \int_{\partial \mathcal{D}_1} \left| \frac{f(x, y) \phi'(x)}{g(x)} \, dx \right|$$

$$+ \frac{2 C_0 C_4 |\bar{g}(y)| e^{-\alpha_2 |\bar{\phi}(y)|}}{\alpha_1 h_1^{n+1}} e^{-\alpha_1 N_1 h_1}$$

(2.16)
$$\le \frac{2 C_0 C_1}{\pi} h_1^{-n} e^{-\frac{\pi d_1}{h_1}} |\bar{g}(y)| e^{-\alpha_2 |\bar{\phi}(y)|} \int_{\partial \mathcal{D}_1} \left| e^{-\alpha_1 |\phi(x)|} \phi'(x) \, dx \right|$$

$$+ \frac{2 C_0 C_4}{\alpha_1 h_1^{n+1}} |\bar{g}(y)| e^{-\alpha_2 |\bar{\phi}(y)|} e^{-\alpha_1 N_1 h_1}$$

$$\le K_1 |\bar{g}(y)| e^{-\alpha_2 |\bar{\phi}(y)|} N_1^{\frac{n+1}{2}} e^{-\sqrt{\pi d_1 \alpha_1 N_1}},$$

where the constant K_1 depends only on $n, g, a_1, b_1, d_1, \alpha_1,$ and h^*.

Using Lemma 2.1(3) again, we get

$$\left| f(x_{k_1}, y) - \sum_{k_2=-N_2}^{N_2} \frac{f(x_{k_1}, y_{k_2})}{\bar{g}_{k_2}} \bar{S}_{k_2}(y) \right|$$

$$\le \frac{2 C_0 C_5}{\alpha_2 h_2} |g(x_{k_1})| e^{-\alpha_1 |\phi(x_{k_1})|} e^{-\alpha_2 N_2 h_2} + \frac{2 C_2}{\pi} e^{-\frac{\pi d_2}{h_2}} \int_{\partial \mathcal{D}_2} \left| \frac{f(x_{k_1}, y) \bar{\phi}'(y)}{\bar{g}(y)} dy \right|$$

$$\le K_2' |g(x_{k_1})| e^{-\alpha_1 |\phi(x_{k_1})|} N_2^{\frac{1}{2}} e^{-\sqrt{\pi d_2 \alpha_2 N_2}},$$

where the constant K_2' depends only on $\bar{g}$, a_2, b_2, d_2, α_2, and h^*. Hence,

$$J_2 \leq \left(\sum_{k_1=-N_1}^{N_1} |S_{k_1}^{(n)}(x)| e^{-\alpha_1 |\phi(x_{k_1})|} \right) K_2' N_2^{\frac{1}{2}} e^{-\sqrt{\pi d_2 \alpha_2 N_2}}.$$

Noting condition (4), and recalling Remark 2.1, we know that

$$|S_{k_1}^{(n)}(x)| \leq C_4 h_1^{-n} \quad \text{for any } k_1 \in \mathbf{Z}.$$

Therefore,

$$\sum_{k_1=-N_1}^{N_1} |S_{k_1}^{(n)}(x)| e^{-\alpha_1 |\phi(x_{k_1})|} \leq C_4 h_1^{-n} \sum_{k_1=-N_1}^{N_1} e^{-\alpha_1 |\phi(x_{k_1})|}$$

$$= C_4 h_1^{-n} \sum_{k_1=-N_1}^{N_1} e^{-\alpha_1 |k_1| h_1} \leq C_4 h_1^{-n} \left[1 + 2 \frac{e^{-\alpha_1 h_1}(1 - e^{-\alpha_1 N_1 h_1})}{1 - e^{-\alpha_1 h_1}} \right]$$

$$\leq C_4 h_1^{-n} \left(1 + \frac{2}{\alpha_1 h_1} \right) \leq C_4' h_1^{-(n+1)}.$$

Thus, we obtain

$$(2.17) \qquad J_2 \leq K_2 N_1^{\frac{(n+1)}{2}} N_2^{\frac{1}{2}} e^{-\sqrt{\pi d_2 \alpha_2 N_2}},$$

where K_2 depends only on n, g, $\bar{g}$, a_j, b_j, d_j, α_j ($j = 1, 2$), and h^*.
   Combining (2.16) and (2.17) completes the proof. □

   In the following we denote $(\partial^{n_1+n_2}/\partial x^{n_1} \partial y^{n_2}) f(x, y)$ by $f^{(n_1, n_2)}(x, y)$.

Theorem 2.5. *Let* $f(x, y)$ *satisfy*
   (1)

$$\left| \frac{f^{(0, n_2)}(x, y)}{g(x)\bar{g}(y)} \right| \leq C_0 e^{-\alpha_1 |\phi(x)|} e^{-\alpha_2 |\bar{\phi}(y)|} \quad \forall (x, y) \in \mathscr{D},$$

$$\left| \frac{f(x, y)}{g(x)\bar{g}(y)} \right| \leq C_0 e^{-\alpha_1 |\phi(x)|} e^{-\alpha_2 |\bar{\phi}(y)|} \quad \forall (x, y) \in \mathscr{D},$$

   (2)

$$\frac{f^{(0, n_2)}(x, y)\phi'(x)}{g(x)} \in B(\mathscr{D}_1'), \qquad \frac{f(x, y)\bar{\phi}'(y)}{\bar{g}(y)} \in B(\mathscr{D}_2'),$$

   (3)

$$\left| \left(\frac{d}{dx} \right)^{n_1} \left\{ \frac{g(x) \sin[\pi \phi(x)/h_1]}{\phi(z) - \phi(x)} \right\} \right| \leq C_1 h_1^{-n_1}$$

for all $x \in \Gamma_1$, $z \in \partial \mathscr{D}_1$, *where* $C_1 = C_1(n_1, g, a_1, b_1)$, *and*

$$\left| \left(\frac{d}{dy} \right)^{n_2} \left\{ \frac{\bar{g}(y) \sin[\pi \bar{\phi}(y)/h_2]}{\bar{\phi}(z) - \bar{\phi}(y)} \right\} \right| \leq C_2 h_2^{-n_2}$$

for all $y \in \Gamma_2$, $z \in \partial \mathscr{D}_2$, *where* $C_2 = C_2(n_2, \bar{g}, a_2, b_2)$,
    (4)

$$|S_0^{(n_1)}(x)| \le C_3 h_1^{-n_1} \quad \forall x \in \Gamma_1, \qquad |\bar{S}_0^{(n_2)}(y)| \le C_4 h_2^{-n_2} \quad \forall y \in \Gamma_2,$$

where C_j $(j = 3, 4)$ *depend only on* β_j, n_j, a_j, b_j, *and* h^*.

Let $h_j = \sqrt{\pi d_j / \alpha_j N_j} \le h^*$, $j = 1, 2$. *Then there exists a constant* C *depending on* g, $\bar{g}$, n_j, β_j, a_j, *and* b_j $(j = 1, 2)$ *such that*

$$\left| f^{(n_1, n_2)}(x, y) - \sum_{k_1 = -N_1}^{N_1} \sum_{k_2 = -N_2}^{N_2} \frac{f_{k_1 k_2}}{g_{k_1} \bar{g}_{k_2}} S_{k_1}^{(n_1)}(x) \bar{S}_{k_2}^{(n_2)}(y) \right|$$

$$\le C N_1^{\frac{n_1 + 1}{2}} \left(E_1 |\bar{g}(y)| e^{-\alpha_2 |\bar{\phi}(y)|} + N_2^{\frac{n_2 + 1}{2}} E_2 \right),$$

where E_j $(j = 1, 2)$ *are defined as in Lemma 2.3.*

Proof. First we have

$$\left| f^{(n_1, n_2)}(x, y) - \sum_{k_1 = -N_1}^{N_1} \sum_{k_2 = -N_2}^{N_2} \frac{f_{k_1 k_2}}{g_{k_1} \bar{g}_{k_2}} S_{k_1}^{(n_1)}(x) \bar{S}_{k_2}^{(n_2)}(y) \right| \le H_1 + H_2,$$

where

$$H_1 = \left| f^{(n_1, n_2)}(x, y) - \sum_{k_1 = -N_1}^{N_1} \frac{f^{(0, n_2)}(x_{k_1}, y)}{g_{k_1}} S_{k_1}^{(n_1)}(x) \right|,$$

$$H_2 = \left| \sum_{k_1 = -N_1}^{N_1} \frac{S_{k_1}^{(n_1)}(x)}{g_{k_1}} \left[f^{(0, n_2)}(x_{k_1}, y) - \sum_{k_2 = -N_2}^{N_2} \frac{f(x_{k_1}, y_{k_2})}{\bar{g}_{k_2}} \bar{S}_{k_2}(y) \right] \right|.$$

Using Lemma 2.1 and Lemma 2.3, we get

$$H_1 \le K_1 |\bar{g}(y)| e^{-\alpha_2 |\bar{\phi}(y)|} N_1^{\frac{n_1 + 1}{2}} e^{-\sqrt{\pi d_1 \alpha_1 N_1}}.$$

Using Lemma 2.1(3) again, we then obtain

$$\left| f^{(0, n_2)}(x_{k_1}, y) - \sum_{k_2 = -N_2}^{N_2} \frac{f(x_{k_1}, y_{k_2})}{\bar{g}_{k_2}} \bar{S}_{k_2}^{(n_2)}(y) \right|$$

$$\le K_2' |g(x_{k_1})| e^{-\alpha_1 |\phi(x_{k_1})|} N_2^{\frac{n_2 + 1}{2}} e^{-\sqrt{\pi d_2 \alpha_2 N_2}},$$

where the constant K_2' depends only on $\bar{g}$, a_2, b_2, d_2, α_2, and h^*. Hence,

$$H_2 \le K_2 N_1^{\frac{(n_1 + 1)}{2}} N_2^{\frac{(n_2 + 1)}{2}} e^{-\sqrt{\pi d_2 \alpha_2 N_2}},$$

where K_2 depends only on n_j, g, $\bar{g}$, a_j, b_j, d_j, α_j $(j = 1, 2)$, and h^*. The proof is completed. $\square$

Remark 2.2. In condition (1), $f^{(0, n_2)}$ can be replaced by $f^{(n_1, 0)}$.

Recalling (2.14) and (2.15), we can improve the result of Theorem 2.5 and get

Theorem 2.6. *Suppose all the conditions in Theorem 2.5 are satisfied. Then*

$$\left| f^{(n_1, n_2)}(x, y) - \sum_{k_1 = -N_1}^{N_1} \sum_{k_2 = -N_2}^{N_2} \frac{f_{k_1 k_2}}{g_{k_1} \bar{g}_{k_2}} S_{k_1}^{(n_1)}(x) \bar{S}_{k_2}^{(n_2)}(y) \right|$$

$$\leq C N_1^{\frac{n_1+1}{2}} \left(E_1 + N_2^{\frac{n_2+1}{2}} E_2 \right).$$

Moreover, if we let

$$(2.18) \qquad N_j = N, \quad d_j = d, \quad \alpha_j = \alpha \qquad (j = 1, 2),$$

then

$$\left| f^{(n_1, n_2)}(x, y) - \sum_{k_1 = -N_1}^{N_1} \sum_{k_2 = -N_2}^{N_2} \frac{f_{k_1 k_2}}{g_{k_1} \bar{g}_{k_2}} S_{k_1}^{(n_1)}(x) \bar{S}_{k_2}^{(n_2)}(y) \right|$$

$$\leq C N^{\frac{n_1+n_2+2}{2}} \exp\left(-\sqrt{\pi d \alpha N} \right) \leq C_1 \exp\left(-C_2 \sqrt{N} \right),$$

where C, C_1, *and* C_2 *are independent of* h *and* N.

Note that the last inequality comes from the well-known formula [1, p. 278]

$$t^k e^{-t} \leq C(\theta) e^{-\theta t}, \qquad \theta \in (0, 1), \ t \geq 0.$$

It should be noted that when we take ϕ, g as in (2.12) and $\bar{\phi}$, $\bar{g}$ as in (2.13), conditions (2)–(4) in Theorem 2.5 or in Theorem 2.6 can be proved to be automatically satisfied, and that the results in these theorems can be improved when $n_1 = n_2 = 0$ [11, Theorem 6.5.2].

3. SOLUTION OF PP

3.1. **Numerical scheme and the Sylvester equation.** Several authors have used Sinc methods to solve the singular Poisson problem PP; for instance, Stenger [9, 10] applied the Sinc-Galerkin method to PP and got a linear system, and Lund et al. [3, 4] got a symmetric linear system by choosing a suitable weight function. The same exponential convergence rate was obtained in all their papers.

Here we use the Sinc-collocation method to solve PP. First, for some non-negative integer k, we set

$$(3.1) \qquad \delta_{j\ell}^{(k)} = h^k \frac{d^k}{d\phi^k} [S(j, h) \circ \phi(z)]|_{z=x_\ell}.$$

By the well-known formula

$$(3.2) \qquad S(k, h) \circ \phi(x) = \frac{1}{2} \int_{-1}^{1} e^{i \frac{\pi}{h} [\phi(x) - kh]t} \, dt,$$

it is easy to verify that

$$(3.3) \qquad \delta_{j\ell}^{(0)} = \begin{cases} 1 & \text{if } j = \ell, \\ 0 & \text{if } j \neq \ell, \end{cases}$$

$$(3.4) \qquad \delta_{j\ell}^{(1)} = \begin{cases} 0 & \text{if } j = \ell, \\ -\dfrac{(-1)^{j-\ell}}{j-\ell} & \text{if } j \neq \ell, \end{cases}$$

$$(3.5) \qquad \delta_{j\ell}^{(2)} = \begin{cases} -\dfrac{\pi^2}{3} & \text{if } j = \ell, \\ -\dfrac{2(-1)^{j-\ell}}{(j-\ell)^2} & \text{if } j \neq \ell, \end{cases}$$

$$(3.6) \qquad \delta_{j\ell}^{(3)} = \begin{cases} 0 & \text{if } j = \ell, \\ -\dfrac{(-1)^{j-\ell}}{(j-\ell)^3}[6 - \pi^2(j-\ell)^2] & \text{if } j \neq \ell, \end{cases}$$

and

$$(3.7) \qquad \delta_{j\ell}^{(4)} = \begin{cases} \dfrac{\pi^4}{5} & \text{if } j = \ell, \\ -\dfrac{(-1)^{j-\ell}}{(j-\ell)^4}[24 - 4\pi^2(j-\ell)^2] & \text{if } j \neq \ell. \end{cases}$$

Now let

$$(3.8) \qquad v_N(x, y) = \sum_{k_1=-N_1}^{N_1} \sum_{k_2=-N_2}^{N_2} \frac{v_{k_1 k_2}}{g_{k_1} \bar{g}_{k_2}} S_{k_1}(x) \bar{S}_{k_2}(y).$$

The coefficients $v_{k_1 k_2}$ ($|k_1| \leq N_1$, $|k_2| \leq N_2$) are determined by requiring that

$$(3.9) \qquad \nabla^2 v_N(x, y)|_{x=x_{i_1}, \, y=y_{i_2}} = f(x_{i_1}, y_{i_2}),$$

where $|i_1| \leq N_1$, $|i_2| \leq N_2$.

Plugging (3.8) into (3.9) gives

$$(3.10) \qquad \begin{aligned} \sum_{k_1=-N_1}^{N_1} \sum_{k_2=-N_2}^{N_2} \frac{v_{k_1 k_2}}{g_{k_1} \bar{g}_{k_2}} &\left(S_{k_1}''(x_{i_1}) \bar{S}_{k_2}(y_{i_2}) + S_{k_1}(x_{i_1}) \bar{S}_{k_2}''(y_{i_2}) \right) \\ &= f_{i_1 i_2} = f(x_{i_1}, y_{i_2}), \qquad |i_1| \leq N_1, \ |i_2| \leq N_2. \end{aligned}$$

Note that (3.10) can be written in matrix-vector form as

$$(3.11) \qquad \mathscr{A} \, \underline{v} = \underline{\mathscr{F}},$$

where

$$\mathscr{A} = \left(\frac{S_{k_1}''(x_{i_1}) \bar{S}_{k_2}(y_{i_2}) + S_{k_1}(x_{i_1}) \bar{S}_{k_2}''(y_{i_2})}{g_{k_1} \bar{g}_{k_2}} \right)$$

and where in the vector $\underline{v} = (v_{k_1 k_2}, \ |k_1| \leq N_1, \ |k_2| \leq N_2)$ the natural ordering of gridpoints from left to right, bottom to top is used, and similarly for the vector $\underline{\mathscr{F}} = (f_{i_1 i_2})$. As indicated in [4], the structure of the matrix $\mathscr{A}$ in the equation (3.11) is inherently vectorizable. We now use another procedure to solve this equation.

It follows from (3.3) that

$$(3.12) \qquad \frac{S_{k_1}(x_{i_1})}{g(x_{k_1})} = \delta_{k_1 i_1}^{(0)} \quad \text{and} \quad \frac{\bar{S}_{k_2}(y_{i_2})}{\bar{g}(y_{k_2})} = \delta_{k_2 i_2}^{(0)},$$

so (3.10) becomes

$$(3.13) \qquad \sum_{k_1=-N_1}^{N_1} \frac{v_{k_1 i_2}}{g_{k_1}} S_{k_1}''(x_{i_1}) + \sum_{k_2=-N_2}^{N_2} \frac{v_{i_1 k_2}}{\bar{g}_{k_2}} \bar{S}_{k_2}''(y_{i_2}) = f_{i_1 i_2}.$$

Let $D_m(g)$ denote the $m \times m$ (where $m = 2N + 1$) diagonal matrix with diagonal elements $g(x_j)$ ($j = -N, \ldots, N$). Set $\tilde{N}_1 = 2N_1 + 1$ and $\tilde{N}_2 = 2N_2 + 1$; then it is not difficult to get the Sylvester equation

$$(3.14) \qquad S_x D_{\tilde{N}_1}\left(\frac{1}{g(x)}\right) U + U D_{\tilde{N}_2}\left(\frac{1}{\bar{g}(y)}\right) S_y^t = W,$$

where S_x is the $\tilde{N}_1 \times \tilde{N}_1$ matrix whose (ik)th entries are $S_k''(x_i)$, and S_y is the $\tilde{N}_2 \times \tilde{N}_2$ matrix whose (ik)th entries are $\bar{S}_k''(y_i)$, respectively. The matrices U and W are $\tilde{N}_1 \times \tilde{N}_2$ with (ik)th entries equal to v_{ik} and f_{ik}, respectively.

Usually, as recommended by Stenger [8] and carried out by Bowers and Lund [2, 4], a procedure to solve (3.14) is as follows. First we choose $g = \phi^{-2}$ and $\bar{g} = \bar{\phi}^{-2}$ ($g = \phi^{-1}$ and $\bar{g} = \bar{\phi}^{-1}$ are chosen in the Sinc-Galerkin method). We know that there exist two nonsingular matrices $E_{\tilde{N}_1 \times \tilde{N}_1}$ and $\bar{E}_{\tilde{N}_2 \times \tilde{N}_2}$ which diagonalize $S_x D_{\tilde{N}_1}(\frac{1}{g(x)})$ and $D_{\tilde{N}_2}(\frac{1}{\bar{g}(y)})S_y^t$, respectively, i.e.,

$$S_x D_{\tilde{N}_1}\left(\frac{1}{g(x)}\right) = E^{-1}\Lambda E, \qquad D_{\tilde{N}_2}\left(\frac{1}{\bar{g}(y)}\right) S_y^t = \bar{E}^{-1}\Lambda'\bar{E},$$

where $\Lambda = (\lambda_i)_{\tilde{N}_1 \times \tilde{N}_1}$ and $\Lambda' = (\lambda_i')_{\tilde{N}_2 \times \tilde{N}_2}$ are the diagonal matrices whose diagonal entries are eigenvalues of $S_x D_{\tilde{N}_1}(\frac{1}{g(x)})$ and $D_{\tilde{N}_2}(\frac{1}{\bar{g}(y)})S_y^t$, respectively. Hence, it is easy to get from (3.14) that $\Lambda Y + Y\Lambda' = X$, where $Y = (y_{ik})_{\tilde{N}_1 \times \tilde{N}_2} = EU\bar{E}^{-1}$ and $X = (x_{ik})_{\tilde{N}_1 \times \tilde{N}_2} = EW\bar{E}^{-1}$. The solution is then

$$(3.15) \qquad Y = \left(\frac{x_{ik}}{\lambda_i + \lambda_k'}\right).$$

This procedure is preferable in view of storage considerations and the ability to handle very singular problems. But it is not very economical computationally since the inverses of E and $\bar{E}$, or some linear systems, need to be computed in order to get X, Y, and U, and also, theoretically, there is a potential difficulty in (3.15) if $\lambda_i + \lambda_k' = 0$.

3.2. **Orthogonalization method.** Here we solve (3.14) by an orthogonalization technique which is computationally much more economical although in theory applicable to a more restrictive class of problems (with respect to the singular behavior of the problems) than the above procedure.

Let $s_{ik} = S_k''(x_i)$. The relation (2.4) yields

$$(3.16) \qquad S_k''(x_i) = g_i'' \delta_{ki}^{(0)} + (2g_i' \phi_i' + g_i \phi_i'') \frac{\delta_{ki}^{(1)}}{h_1} + g_i \phi_i'^2 \frac{\delta_{ki}^{(2)}}{h_1^2}.$$

Taking $g = \phi'^{-\frac{1}{2}}$ and using the identity [4]

$$(3.17) \qquad \phi'^{-\frac{3}{2}} (\phi'^{-\frac{1}{2}})'' = -\frac{1}{4},$$

we get

$$s_{ik} = \frac{1}{h_1^2} \phi_i'^{\frac{3}{2}} \left(\delta_{ki}^{(2)} - \frac{1}{4} h_1^2 \delta_{ki}^{(0)} \right).$$

If the choice $\bar{g} = \bar{\phi}'^{-\frac{1}{2}}$ is made, we can also write

$$\bar{s}_{ik} = \bar{S}_k''(y_i) = \frac{1}{h_2^2} \bar{\phi}_i'^{\frac{3}{2}} \left(\delta_{ki}^{(2)} - \frac{1}{4} h_2^2 \delta_{ki}^{(0)} \right).$$

Now we let $I_m^{(k)}$ be the $m \times m$ matrices $(m = 2N + 1)$ whose (jl)th entry is $\delta_{jl}^{(k)}$, given by (3.3)–(3.7). Then the equation (3.14) takes the form

$$(3.18) \qquad A_x Y + Y A_y = h^2 W',$$

where

$$(3.19) \qquad \begin{aligned} A_x &= D_{\tilde{N}_1} \left(\phi'(x) \right) A_1 D_{\tilde{N}_1} \left(\phi'(x) \right), \\ A_y &= D_{\tilde{N}_2} \left(\bar{\phi}'(y) \right) A_2 D_{\tilde{N}_2} \left(\bar{\phi}'(y) \right), \end{aligned}$$

$$(3.20) \qquad \begin{aligned} Y &= D_{\tilde{N}_1}(\phi'^{-\frac{1}{2}}(x)) U D_{\tilde{N}_2}(\bar{\phi}'^{-\frac{1}{2}}(y)), \\ W' &= D_{\tilde{N}_1}(\phi'^{-\frac{1}{2}}(x)) W D_{\tilde{N}_2}(\bar{\phi}'^{-\frac{1}{2}}(y)), \end{aligned}$$

and

$$(3.21) \quad A_j = (a_{ik})_{\tilde{N}_j \times \tilde{N}_j} = (\delta_{ki}^{(2)} - \tfrac{1}{4} h_j^2 \delta_{ki}^{(0)}) = I_{\tilde{N}_j}^{(2)} - \tfrac{1}{4} h_j^2 I_{\tilde{N}_j}^{(0)}, \qquad j = 1, 2.$$

It should be pointed out that the system (3.18) is identical with the system obtained by using an entirely different procedure (the weighted Sinc-Galerkin method) [4]. There are few classes of basis functions where the collocation method and the Galerkin method yield the same discrete system. Typically, the former is advertised as the one with the simpler assembly, while the latter is usually more accurate. Here, the Sinc function gives the best of both worlds.

Lemma 3.1. *Both matrices A_x and A_y are symmetric negative definite.*

The proof of this lemma is easy if we notice that the A_j, given by (3.21), are symmetric negative definite [8] and the diagonal matrices $D_{\tilde{N}_1}(\phi'(x))$ and $D_{\tilde{N}_2}(\bar{\phi}'(y))$ are nonsingular.

Lemma 3.2. *If $h_1 = h_2 = h$ and $N_1 = N_2 = N$, then $A_y = \kappa^2 A_x$, where $\kappa = (b_1 - a_1)/(b_2 - a_2)$.*

Proof. It is easy to show from (2.12) and (2.13) that

$$\phi_i' = \frac{(1 + e^{ih})^2}{(b_1 - a_1)e^{ih}} \quad \text{and} \quad \bar{\phi}_i' = \frac{(1 + e^{ih})^2}{(b_2 - a_2)e^{ih}}.$$

Thus, $D_N\left(\bar{\phi}'(y)\right) = \kappa D_N\left(\phi'(x)\right)$. The relationships $N_1 = N_2 = N$ yield $A_1 = A_2 = A$. The proof is then completed by using the formulas (3.19). □

From now on, we shall choose $h_1 = h_2 = h$ and $N_1 = N_2 = N$. Thus, all of the matrices in (3.18) are $m \times m$ $(m = 2N + 1)$, so the subscripts on the matrices are omitted in the rest of the paper.

The symmetry of the matrix A_x implies that there exists an orthogonal matrix Q with the property that $A_x = Q\Lambda Q^t$, where $\Lambda = (\lambda_i)$ is the diagonal matrix with the eigenvalues of A_x as diagonal entries. It is obvious that $A_y = \kappa^2 Q\Lambda Q^t$. Hence, (3.18) becomes

(3.22) $$\Lambda Y' + \kappa^2 Y'\Lambda = h^2 Z,$$

where $Y' = (y'_{ik}) = Q^t Y Q$ and $Z = (z_{ik}) = Q^t W' Q$. The solution of (3.22) is then

(3.23) $$Y' = (y'_{ik}) = \left(\frac{h^2 z_{ik}}{\lambda_i + \kappa^2 \lambda_k}\right).$$

Lemma 3.1 implies $\lambda_i + \kappa^2 \lambda_k < 0$, so the potential difficulty appearing in (3.15) is avoided.

To calculate the original U from $D(\phi'^{\frac{1}{2}}(x))QY'Q^tD(\bar{\phi}'^{\frac{1}{2}}(y))$ requires only matrix-matrix products, which can be easily done in parallel.

3.3. Computational results.
The Sinc-collocation method with orthogonalization has been tested on a large family of problems, both analytic and singular. As is indicated in [2], one of the advantages of the Sinc method is that it automatically determines the graded mesh. So there is no modification in the discrete system (3.18) when it passes from analytic problems to singular problems. All the following three examples are singular, and the second derivatives of their true solutions are undefined on the boundary.

The discussion in the previous section leads us to recommend the following parameter selections. The positive parameter α should be chosen so that the true solution $u(x, y)$ satisfies the decay condition

$$|u(x, y)| \leq C((x - a)(b - y))^{\alpha + 1/2}.$$

Then the parameter h is given by $h = (\frac{\pi d}{\alpha N})^{\frac{1}{2}}$, where the angle d is taken to be $\frac{\pi}{2}$ for the problems considered.

The maximum absolute error between the numerical approximation $v_N(x, y)$ and the true solution $u(x, y)$ at the Sinc grid points is defined as $\|E\|_g$ and the maximum absolute error at the 100 equally spaced points is defined as $\|E\|_u$, where these equally spaced points (x_i, y_j) are chosen to be $x_i = a_1 + \frac{i}{100}(b_1 - a_1)$ and $y_j = a_2 + \frac{j}{100}(b_2 - a_2)$. The predicted asymptotic convergence rate is $O(e^{-(\pi d \alpha N)^{1/2}})$. The code was run in double precision using the

FORTRAN-77 compiler on a DECstation/UNIX computer. The numerical results for different N are shown in the following tables.

Example 3.1.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y), \qquad u(x, y) = (1 - x^2)^{\frac{7}{4}}(y - 1)^2(4 - y)^2,$$

where $\Omega = (-1, 1) \times (1, 4)$. The parameter α is chosen to be $\alpha = 5/4$. The result is shown in Table 1.

TABLE 1

N	h	$\|E\|_g$	$\|E\|_u$	asymptotic error
2	1.404963	1.0802D-1	1.1046D-1	2.9825D-2
4	0.993459	1.1282D-2	1.1887D-2	6.9620D-3
8	0.702481	1.0286D-3	1.0048D-3	8.8953D-4
16	0.496729	2.4315D-5	2.3147D-5	4.8469D-5
32	0.351241	9.5987D-8	9.2376D-8	7.9127D-7

Example 3.2.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y), \qquad u(x, y) = (x^{3/2} - x)(y^{3/2} - y),$$

where $\Omega = (0, 1) \times (0, 1)$. This example is found in [2], where the authors used it to illustrate the ability of the Sinc method to solve singular problems, and where they compared it with the difficulties inherent in the standard finite difference or finite element methods presented in [12]. The parameter α is here chosen to be $\alpha = 1/2$. The result is shown in Table 2.

TABLE 2

N	h	$\|E\|_g$	$\|E\|_u$	asymptotic error
2	2.221441	2.5315D-4	7.4670D-4	1.0845D-1
4	1.570796	7.0055D-5	1.2511D-4	4.3214D-2
8	1.110721	7.7647D-6	1.3159D-5	1.1762D-2
16	0.785398	2.5963D-7	5.1589D-7	1.8674D-3
32	0.555360	1.7498D-9	8.6613D-9	1.3834D-4

Example 3.3.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y), \qquad u(x, y) = xy \ln x \ln y,$$

where $\Omega = (0, 1) \times (0, 1)$. This example is also found in [2] and illustrates a logarithmic singularity. The parameter α can be chosen as large as possible in this case, i.e., $\alpha = 1/2$. The result is shown in Table 3.

TABLE 3

N	h	$\|E\|_g$	$\|E\|_u$	asymptotic error
2	2.221441	3.9517D-3	8.1485D-3	1.0845D-1
4	1.570796	1.3048D-3	2.0009D-3	4.3214D-2
8	1.110721	1.8299D-4	2.6689D-4	1.1762D-2
16	0.785398	8.5641D-6	1.7600D-5	1.8674D-3
32	0.555360	7.9693D-8	5.7605D-7	1.3834D-4

4. SOLUTION OF FOURTH-ORDER POISSON-LIKE PROBLEMS

We now consider the fourth-order Poisson-like problem:

$$
(4.1) \quad \frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} = f(x, y), \qquad (x, y) \in \Omega \equiv (a_1, b_1) \times (a_2, b_2),
$$

$$
u|_{\partial\Omega} = \left.\frac{\partial u}{\partial x}\right|_{\partial\Omega} = \left.\frac{\partial u}{\partial y}\right|_{\partial\Omega} = 0.
$$

Smith et al. [6] have discussed the one-dimensional case of the above problem by using the Sinc-Galerkin method and got an accurate numerical solution to an order of $O(e^{-\sqrt{\pi d \alpha N}})$.

Here we can still solve (4.1) by using the Sinc-collocation method with the orthogonalization technique described in §3.2.

Let $v_N(x, y)$ be defined by (3.8) with the choice $h_1 = h_2 = h$ and $N_1 = N_2 = N$, where $v_{k_1 k_2}$ will be determined by

$$
(4.2) \quad \sum_{k_1=-N}^{N} \frac{v_{k_1 i_2}}{g_{k_1}} S_{k_1}''''(x_{i_1}) + \sum_{k_2=-N}^{N} \frac{v_{i_1 k_2}}{\bar{g}_{k_2}} \bar{S}_{k_2}''''(y_{i_2}) = f_{i_1 i_2}.
$$

Analogous to §3.2, a matrix representation of the equation (4.2) can be obtained in the form

$$
(4.3) \quad S_x D\left(\frac{1}{g(x)}\right) U + U D\left(\frac{1}{\bar{g}(y)}\right) S_y^t = W,
$$

where S_x and S_y are $m \times m$ matrices whose (ik)th entries are $S_k''''(x_i)$ and $\bar{S}_k''''(y_i)$, respectively. The matrices U and W are also $m \times m$, with (ik)th entries given by v_{ik} and f_{ik}, respectively.

It is easy to verify from (2.4) that

$$
S_k''''(x_i) = g_i''''\delta_{ki}^{(0)} + \frac{1}{h}a_i^{(1)}\delta_{ki}^{(1)} + \frac{1}{h^2}a_i^{(2)}\delta_{ki}^{(2)} + \frac{1}{h^3}a_i^{(3)}\delta_{ki}^{(3)} + \frac{1}{h^4}g_i\phi_i'^4\delta_{ki}^{(4)},
$$

where

$$
a^{(1)} = 4g'''\phi' + 6g''\phi'' + 4g'\phi''' + g\phi'''',
$$

$$
a^{(2)} = 6g''\phi'^2 + 12g'\phi'\phi'' + 3g\phi''^2 + 4g\phi'\phi''',
$$

$$
a^{(3)} = 4g'\phi'^3 + 6g\phi'^2\phi'',
$$

and the $\delta_{ki}^{(j)}$, $j = 0, \dots, 4$ are defined in (3.3)–(3.7).

Take

$$
(4.4) \quad g(x) = \left(\phi'(x)\right)^{-\frac{3}{2}};
$$

then we get $a^{(3)}(x) = 0$. Direct verification, using (3.17), gives $a^{(1)}(x) = 0$ and $a^{(2)}(x) = -\frac{5}{2}\phi'^{\frac{5}{2}}(x)$. It should be noted that, by using (4.4) and (3.17), we can prove that

$$g'''' = \frac{9}{16}\phi'^{\frac{5}{2}}.$$

It then follows that

(4.5) $S_k''''(x_i) = \frac{1}{h^4}\phi_i'^{\frac{5}{2}}\left\{\delta_{ki}^{(4)} - \frac{5}{2}h^2\delta_{ki}^{(2)} + \frac{9}{16}h^4\delta_{ki}^{(0)}\right\}.$

Similarly, we get

(4.6) $\bar{S}_k''''(y_i) = \frac{1}{h^4}\bar{\phi}_i'^{\frac{5}{2}}\left\{\delta_{ki}^{(4)} - \frac{5}{2}h^2\delta_{ki}^{(2)} + \frac{9}{16}h^4\delta_{ki}^{(0)}\right\}$

by taking

(4.7) $\bar{g}(y) = \left(\bar{\phi}'(y)\right)^{-\frac{3}{2}}.$

Using the same procedure as in §3.2, we easily obtain

(4.8) $A_x Y + Y A_y = h^4 W'.$

But now

(4.9) $A_x = D(\phi'^2(x))AD(\phi'^2(x)), \qquad A_y = D(\bar{\phi}'^2(y))AD(\bar{\phi}'^2(y)),$

(4.10)
$$Y = D(\phi'^{-\frac{1}{2}}(x))UD(\bar{\phi}'^{-\frac{1}{2}}(y)),$$
$$W' = D(\phi'^{-\frac{1}{2}}(x))WD(\bar{\phi}'^{-\frac{1}{2}}(y)),$$

and

(4.11) $A = (a_{ik}) = (\delta_{ki}^{(4)} - \frac{5}{2}h^2\delta_{ki}^{(2)} + \frac{9}{16}h^4\delta_{ki}^{(0)}) = I^{(4)} - \frac{5}{2}h^2I^{(2)} + \frac{9}{16}h^4I^{(0)}.$

Since the matrix $I^{(4)}$ is positive definite [6], we can show, in the same way as before, that both of the matrices A_x and A_y are symmetric positive definite, and furthermore, under the assumption that $h_1 = h_2 = h$, $N_1 = N_2 = N$, that we have $A_y = \kappa^4 A_x$. Hence, the same procedure of our orthogonalization technique described in §3.2 can be applied to the Poisson-like problem (4.1), except that the matrix Y' in the equation (3.23) must be replaced by

$$Y' = (y_{ik}') = \left(\frac{h^4 z_{ik}}{\lambda_i + \kappa^4\lambda_k}\right).$$

Also, the positive parameter α should be selected to satisfy

$$|u(x, y)| \le C((x-a)(b-y))^{\alpha+\frac{3}{2}}.$$

Here we give some numerical results, where $\|E\|_g$, $\|E\|_u$, and d are defined as before.

Example 4.1.

$$\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} = f(x, y), \qquad u(x, y) = (1 - x^2)^3(y - 1)^3(4 - y)^3,$$

where $\Omega = (-1, 1) \times (1, 4)$. This is a problem whose solution is known an-
alytically. The parameter α is chosen to be $\alpha = 3/2$. The result is shown in
Table 4.

TABLE 4

N	h	$\|E\|_g$	$\|E\|_u$	asymptotic error
2	1.282550	9.4962D-1	9.5118D-1	2.1330D-2
4	0.906900	2.7328D-2	5.9729D-2	4.3334D-3
8	0.641275	3.0343D-3	3.7935D-3	4.5496D-4
16	0.453450	7.3851D-5	7.3851D-5	1.8779D-5
32	0.320637	9.3745D-6	9.4706D-6	2.0699D-7

Example 4.2.

$$\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} = f(x, y), \qquad u(x, y) = y^3 (\sin x)^3 \ln^3 y,$$

where $\Omega = (0, \pi) \times (0, 1)$. This problem is singular in y. The parameter α
is selected to be $\alpha = 3/2$. The result is shown in Table 5.

TABLE 5

N	h	$\|E\|_g$	$\|E\|_u$	asymptotic error
2	1.282550	5.8964D-3	8.3871D-3	2.1330D-2
4	0.906900	2.8201D-3	2.9605D-3	4.3334D-3
8	0.641275	5.4495D-4	5.4531D-4	4.5496D-4
16	0.453450	2.9301D-5	2.9605D-5	1.8779D-5
32	0.320637	2.5942D-7	2.6026D-7	2.0699D-7

Example 4.3.

$$\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} = f(x, y), \qquad u(x, y) = (x(1 - x))^{\frac{7}{2}} (y(2 - y))^{\frac{10}{3}},$$

where $\Omega = (0, 1) \times (0, 2)$. This example illustrates the highest degree of singu-
larity (u has fourth partial derivatives which are undefined on the boundary.)
The parameter α is chosen to be $\alpha = 11/6$. The result is shown in Table 6.

TABLE 6

N	h	$\|E\|_g$	$\|E\|_u$	asymptotic error
2	1.160110	8.5673D-4	8.5673D-4	1.4211D-2
4	0.820322	1.3176D-5	3.4646D-5	2.4402D-3
8	0.580055	1.0888D-6	1.4810D-6	2.0196D-4
16	0.410161	1.5501D-8	1.6013D-8	5.9544D-6
32	0.290027	3.4867D-10	3.4871D-10	4.0786D-8

ACKNOWLEDGMENTS

BIBLIOGRAPHY

1. E. P. Doolan, J. J. H. Miller and W. H. A. Schilders, *Uniform numerical methods for problems with initial and boundary layers,* Boole Press, Dublin, 1980.

2. K. L. Bowers and J. Lund, *Numerical solution of singular Poisson problems via the Sinc-Galerkin method,* SIAM J. Numer. Anal. **24** (1987), 36–51.

3. J. Lund, *Symmetrization of the Sinc-Galerkin method for boundary value problems,* Math. Comp. **47** (1986), 571–588.

4. J. Lund, K. L. Bowers, and K. M. McArthur, *Symmetrization of the Sinc-Galerkin method with block techniques for elliptic equations,* IMA J. Numer. Anal. **9** (1989), 29–46.

5. L. Lundin and F. Stenger, *Cardinal-type approximations of a function and its derivatives,* SIAM J. Math. Anal. **10** (1979), 139–160.

6. R. C. Smith, K. L. Bowers, and J. Lund, *Efficient numerical solution of fourth-order problems in the modeling of flexible structures,* Computation and Control (K. Bowers and J. Lund, eds.), Birkhäuser, Boston, 1988, pp. 283–297.

7. F. Stenger, *Integration formulas via the trapezoidal formula,* J. Inst. Math. Appl. **12** (1973), 103–114.

8. _____, *A "Sinc-Galerkin" method of solution of boundary value problems,* Math. Comp. **33** (1979), 85–109.

9. _____, *Numerical methods based on Whittaker cardinal, or Sinc functions,* SIAM Rev. **23** (1981), 165–224.

10. _____, *Sinc methods of approximate solution of partial differential equations,* Advances in Computer Methods for Partial Differential Equations, Philadelphia, 1984, pp. 244–251.

11. _____, *Numerical methods based on Sinc and analytic functions,* Springer-Verlag, New York, 1993.

12. A. Weiser, S. C. Eisenstat, and M. H. Schultz, *On solving elliptic problems to moderate accuracy,* SIAM J. Numer. Anal. **17** (1980), 908–929.

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